$S-2S$を考える

$$S = 1\cdot 1 + 2\cdot 2^1 + 3\cdot 2^2 + \cdots + n2^{n-1}$$

$$2S = \qquad\qquad 1\cdot 2^1 + 2\cdot 2^2 + \cdots + (n-1)2^{n-1} + n2^n$$

$$\therefore S - 2S = 1 + 2 + 2^2 + \cdots 2^{n-1} - n2^n = \left(\sum_{k=1}^{n} 2^{k-1}\right) - n2^n$$

$$= \left\{\sum_{k=1}^{n} (2-1)2^{k-1}\right\} - n2^n = \left(\sum_{k=1}^{n} 2^k - 2^{k-1}\right) - n2^n$$

$$= 2^n - 2^0 - n2^n = (1-n)2^n - 1$$

$$\therefore S = (n-1)2^n + 1$$