$(1) A+B+C=180^\circ \wedge \angle A=60^\circ$より、$B+C=120^\circ$

$\therefore C=120^\circ-B$としてCを消去する

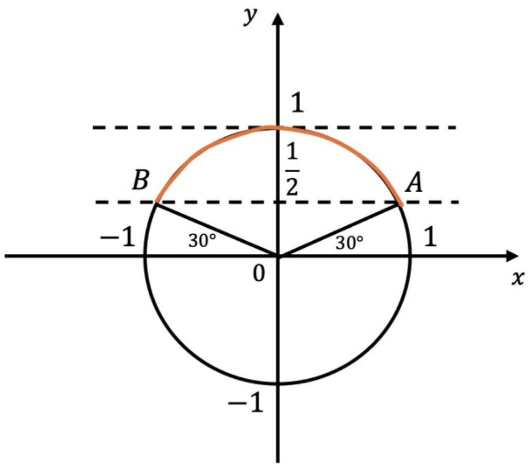
$$\Longrightarrow \sin B+\sin C=\sin B+\sin(120^\circ-B)$$

$$=\sin B+\sin 120^\circ\cos B-\sin B\cos 120^\circ$$

$$=\sin B+\frac{\sqrt{3}}{2}\cos B-\left(-\frac{1}{2}\right)\sin B=\frac{3}{2}\sin B+\frac{\sqrt{3}}{2}\cos B$$

$$=\sqrt{3}\left(\frac{\sqrt{3}}{2}\sin B+\frac{1}{2}\cos B\right)=\sqrt{3}\sin(B+30^\circ)$$

ここで、$0<B<120^\circ$であるから $30^\circ<\ B+30^\circ<150^\circ$となる



すると左図の$A\to B$を動くのがわかる

$$\therefore \frac{1}{2}<\sin(B+30^\circ)\le 1$$

$$\Leftrightarrow \frac{\sqrt{3}}{2}<\sqrt{3}\sin(B+30^\circ)\le\sqrt{3}$$

$$\Leftrightarrow \frac{\sqrt{3}}{2}<\sin B+\sin C\le\sqrt{3}$$

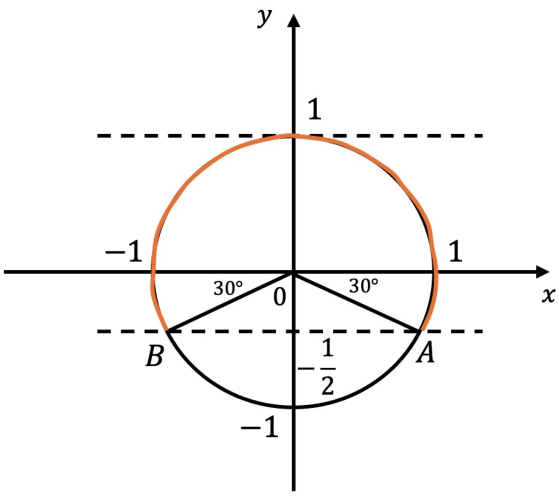
$$(2)\ \sin B\sin C=\sin B\sin(120^\circ-B)$$

$$=\sin B\,(\sin 120^\circ\cos B-\sin B\cos 120^\circ)=\frac{\sqrt{3}}{2}\sin B\cos B+\frac{1}{2}(\sin B)^2$$

$$=\frac{\sqrt{3}}{4}\sin 2B+\frac{1}{4}(1-\cos 2B)=\frac{1}{2}\left(\frac{\sqrt{3}}{2}\sin 2B-\frac{1}{2}\cos 2B\right)+\frac{1}{4}$$

$$=\frac{1}{2}\sin(2B-30^\circ)+\frac{1}{4}$$

この時、$0<B<120^\circ\Leftrightarrow 0<2B<240^\circ\Leftrightarrow -30^\circ<2B-30^\circ<210^\circ$



よって、左図より

$$-\frac{1}{2} < \sin(2B - 30°) \leq 1$$

$$\iff 0 < \frac{1}{2}\sin(2B - 30°) + \frac{1}{4} \leq \frac{3}{4}$$

$$\iff 0 < \sin B \sin C \leq \frac{3}{4}$$