$$\int_0^1 x \log(x^2 + 1)\,dx = \int_0^1 \left(\frac{x^2}{2}\right)' \log(x^2 + 1)\,dx$$

$$= \left[\frac{x^2}{2}\log(x^2 + 1)\right]_0^1 - \int_0^1 \frac{x^2}{2} \cdot \frac{2x}{x^2 + 1}\,dx$$

$$= \frac{1}{2}\log 2 - \int_0^1 \frac{x^3}{x^2 + 1}\,dx$$

$$= \frac{1}{2}\log 2 - \int_0^1 \frac{x(x^2 + 1) - x}{x^2 + 1}\,dx$$

$$= \frac{1}{2}\log 2 - \int_0^1 \left(x - \frac{x}{x^2 + 1}\right) dx = \frac{1}{2}\log 2 - \left[\frac{x^2}{2} - \frac{1}{2}\log(x^2 + 1)\right]_0^1$$

$$= \frac{1}{2}\log 2 - \frac{1}{2} + \frac{1}{2}\log 2$$

$$= -\frac{1}{2} + \log 2$$